

# Chern-Simons Theory of Fractional Quantum Hall Effect in (Pseudo) Massless Dirac Electrons

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## Abstract

We derive the effective field theory from the microscopic Hamiltonian of interacting two-dimensional (pseudo) Dirac electrons by performing a statistic gauge transformation. The quantized Hall conductance are expected to be  $\sigma_{xy} = \frac{e^2}{h}(2k - 1)$  with  $k$  is arbitrary integer. There are also topological excitations which have fractional charge and obey fractional statistics.

## 1 Introduction

The discovery of the fractional quantum hall effect[1] in GaAs have deepen our understanding of quantum many-body system very much.[2] The completely new matter state in FQHE is called quantum hall liquid which opened a new chapter of condensed matter physics. The quantum hall liquid has a new kind of order (topological order) that is beyond the Landau symmetry-breaking description.[3] After laughlin's first successful theory in which the famous laughlin wave function was proposed[4]. A completely and first principle construction of the effective-field-theory was given by Zhang, Hansson and Kivelson[5, 7] and later extended by Lee and Zhang[6]. In the building of the theory, a singular gauge transformation is used to map the interacting fermions problem to one of interacting bosons coupled to an additional gauge field (the Chern-Simons field).[7, 8] This successful effective-field-theory not only explains the experimental facts completely, but also demonstrates that there is a deep connection between superfluid and FQHE. Until now, the FQHE is only observed in GaAs, in which the electrons are conventional ("non-relativistic"). A natural question is that what is the properties of the possible FQHE in (pseudo) Massless Dirac electrons that exist in Graphene.[20] The possible FQHE in Graphene has been discussed in many papers[9, 10, 11, 12, 13], and until now, a effective field theory for FQHE in "relativistic" electrons system is still needed . In section

2, we try to derive the effective-field-theory of FQHE from the full spin polarized interacting microscopic massless Dirac fermions Hamiltonian by using the same statistic gauge transformation in ref. 7. Then we consider the uniform mean-field solution and the topological excitations of the theory in section 3 and section 4 respectively. Finally, conclusion is given in section 5.

## 2 The Derivation of Chern-Simons Theory

We begin by writing down the hamiltonian of interacting two dimensional massless fermions [14]

$$H = \sum_i v_F [\vec{\sigma} \cdot (\mathbf{p}_i - \frac{e}{c} \mathbf{A}(\mathbf{r}_i))] + \sum_i e A_0(\mathbf{r}_i) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|). \quad (1)$$

$v_F$  is the fermi speed. We have assumed the (pseudo) Dirac electrons are completely polarized. This Hamiltonian is different from the one in Graphene, since there are two kinds of (pseudo) spin while we only consider one kind here. The space wave function of the Hamiltonian  $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$  must be totally antisymmetric. The eigenvalue equation of the hamiltonian reads

$$H\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = E\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N). \quad (2)$$

$\Psi$  is a two components wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \begin{bmatrix} \Psi_1(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \\ \Psi_2(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \end{bmatrix}. \quad (3)$$

Both components are totally antisymmetric. This N-fermion problem can be mapped to a bosonic eigenvalue one with symmetric wave function by performing a statistic gauge transformation.[5, 7]

$$H'\Psi'(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = E\Psi'(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N), \quad (4)$$

$\Psi'(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$  has two components  $\Psi'_1, \Psi'_2$ , both are completely symmetric when we exchange the places of any two particles. In which

$$H' = U^{-1} H U, \Psi' = U^{-1} \Psi, \quad (5)$$

$$U = I \exp(-i \sum_{i < j} \frac{\theta}{\pi} \alpha_{ij}). \quad (6)$$

$I$  is the two dimensional identity matrix.  $\alpha_{ij}$  is the angel between the x-axis and the the vector  $\mathbf{r}_i - \mathbf{r}_j$ .  $\theta = (2k+1)\pi$  must be satisfied to guarantee  $\Psi'$  is symmetric. By introducing the statistic gauge operator

$$\mathbf{a}(\mathbf{r}) = \frac{\phi_0 \theta}{2\pi^2} \sum_{i \neq j} \nabla \alpha_{ij}, \quad (7)$$

$\phi_0 = \frac{2\pi}{e}$  is the flux quantum. By performing the statistic gauge transformation (5). We can write the new bosonic Hamiltonian

$$H' = \sum_i v_F [\vec{\sigma} \cdot (\mathbf{p}_i - \frac{e}{c} \mathbf{A}(\mathbf{r}_i) - \frac{e}{c} \mathbf{a}(\mathbf{r}_i))] + \sum_i e A_0(\mathbf{r}_i) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|). \quad (8)$$

The physical meaning of the introduced statistic gauge field  $\mathbf{a}$  can be seen from the statistic gauge transformation:

$$\exp(i \sum_{i < j} \frac{\theta}{\pi} \alpha_{ij}) \mathbf{p}_i \exp(-i \sum_{i < j} \frac{\theta}{\pi} \alpha_{ij}) = \mathbf{p}_i - \frac{e}{c} \frac{\phi_0 \theta}{2\pi^2} \sum_{i \neq j} \nabla \alpha_{ij} = \mathbf{p}_i - \frac{e}{c} \mathbf{a}, \quad (9)$$

$\mathbf{a}$  describes the gauge interaction between particles.

In the language of second quantization, the Hamiltonian can be rewritten as

$$H' = \int d^2 \mathbf{r} \phi^+(\mathbf{r}) [e A_0 + v_F (\vec{\sigma} \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}) - \frac{e}{c} \mathbf{a}(\mathbf{r})))] \phi(\mathbf{r}) + \frac{1}{2} \int d^2 \mathbf{r} \int d^2 \mathbf{r}' (\rho(\mathbf{r}) - \bar{\rho}) V(\mathbf{r} - \mathbf{r}') (\rho(\mathbf{r}') - \bar{\rho}), \quad (10)$$

where  $\rho(\mathbf{r}) = \phi^+(\mathbf{r})\phi(\mathbf{r})$  is the particle density at  $\mathbf{r}$ .  $\bar{\rho}$  is the average particle density. The statistic operator expressed in second quantization reads

$$a^\alpha = \frac{\phi_0 \theta}{2\pi^2} \varepsilon^{\alpha\beta} \int d^2 \mathbf{r}' \frac{\mathbf{r}^\beta - \mathbf{r}'^\beta}{|\mathbf{r} - \mathbf{r}'|^2} \rho(\mathbf{r}'). \quad (11)$$

$\varepsilon^{\alpha\beta} = \varepsilon^{0\alpha\beta}$  is the standard Levi-Civita tensor. From (11) we know that  $\mathbf{a}$  is decided by  $\rho(\mathbf{r}) = \phi^+(\mathbf{r})\phi(\mathbf{r})$ , it is not an independent dynamic quantity. We must know the equation of motion for  $\mathbf{a}$  in our theory to get the effective action of the system. In the Coulomb gauge, (11) is the solution of

$$\varepsilon^{\alpha\beta} \partial_\alpha a_\beta(\mathbf{r}) = \phi_0 \frac{\theta}{\pi} \rho(\mathbf{r}). \quad (12)$$

Equation just give  $\mathbf{a}(\mathbf{r})$  at a given time. In order to get the dynamics of the statistic gauge field, we take the time derivation of (12)

$$\varepsilon^{\alpha\beta} \partial_\alpha \dot{a}_\beta(\mathbf{r}) = \phi_0 \frac{\theta}{\pi} \dot{\rho}(\mathbf{r}). \quad (13)$$

By using the continuity equation  $\partial_t \rho(\mathbf{r}, t) + \partial_\alpha j^\alpha(\mathbf{r}, t) = 0$ , we have

$$\varepsilon^{\alpha\beta} \dot{a}_\beta(\mathbf{r}) = -\phi_0 \frac{\theta}{\pi} j^\alpha. \quad (14)$$

Equation (12) and (14) together give the equations of motion for  $\mathbf{a}$ , they can be derived from the Chern-Simons lagrange

$$\mathcal{L} = \frac{1}{2} \frac{e\pi}{\phi_0 \theta} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - a_\mu j^\mu. \quad (15)$$

Then we can formulate the problem in coherent state path integral, all the thermodynamic properties and the electromagnetic response of the system is completely contained in the following partition function  $Z$

$$Z[A_\mu] = \int [da_\mu][d\phi] \exp(iS[a_\mu] + iS[\phi]). \quad (16)$$

In which

$$S[a_\mu] = \int dt \int d^2\mathbf{r} \frac{\pi e}{2\theta\phi_0} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad (17)$$

$$S[\phi] = \int dt \int d^2\mathbf{r} \phi^\dagger[\mathbf{r}] (i\partial_t - \frac{e}{c}(A_0 + a_0) - \vec{\sigma} \cdot (\mathbf{p} - \frac{e}{c}(\mathbf{A} + \mathbf{a}))) \phi(\mathbf{r}) - \int dt \int d^2\mathbf{r} \int d^2\mathbf{r}' \delta\rho(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \delta\rho(\mathbf{r}). \quad (18)$$

$\phi$  also has two components. It is clear that the action has a local U(1) gauge symmetry

$$\phi \rightarrow \phi' = (I \exp(i\gamma)) \phi, a_\mu \rightarrow a'_\mu = a_\mu - \frac{1}{e} \partial_\mu \gamma. \quad (19)$$

$I$  is the two dimensional unit matrix.

### 3 The Mean Field Theory

First we consider the mean-field solution of the system when there is no external electric field ( $A_0 = 0$ ). For the perpendicular external magnetic field along the  $z$  axis,  $\varepsilon^{\alpha\beta} \partial_\alpha A_\beta = -B$ . We can guess the mean-field solution

$$\phi(\mathbf{r}) = \begin{pmatrix} \sqrt{\bar{\rho}_1} \\ \sqrt{\bar{\rho}_2} \end{pmatrix}, a(\mathbf{r}) = -A(\mathbf{r}), a_0(\mathbf{r}) = 0. \quad (20)$$

Both  $\bar{\rho}_1$  and  $\bar{\rho}_2$  are constant, they are not independent of each other since  $\bar{\rho}_1 + \bar{\rho}_2 = \bar{\rho}$  is the average electron density. It is easy to prove that the solution satisfy all the equations of motion derived from the action. Further more, the statistic gauge field is related to the particle density via (12), we get

$$B = \phi_0 \frac{\theta}{\pi} \bar{\rho}, \quad (21)$$

which means that the filling factor

$$\nu = \frac{\pi}{\theta} = \frac{1}{2k-1}. \quad (22)$$

In this special state, we can consider a bosonic system without a magnetic field where a Bose condensation or superfluid will happen.

Then we calculate the Hall conductance, we apply an external scalar potential  $A_0$  with  $\partial_\mu A_0 = -E_\mu$ . The gauge invariant current

$$\langle j_\alpha(\mathbf{r}) \rangle = \left\langle \frac{\delta S}{\delta A_\alpha} \right\rangle. \quad (23)$$

From the action, we have

$$j_\alpha(\mathbf{r}) = \frac{\delta S}{\delta A_\alpha} = \frac{\delta S_\phi}{\delta a_\alpha} = -\frac{\delta S_a}{\delta a_\alpha}. \quad (24)$$

In the last step of (24) we have used the static field equation that  $\frac{\delta S}{\delta a_\alpha} = 0$ . After integration by parts the Chern-Simons lagrangian can be written as

$$\mathcal{L}_a = \frac{e\pi}{2\theta\phi_0} \epsilon^{\alpha\beta} (2a_\alpha \partial_\beta a_0 - a_\alpha \partial_t a_\beta). \quad (25)$$

Since the statistic gauge field  $\mathbf{a}$  is static, we finally get

$$j_\alpha = \frac{e^2\pi}{h\theta} \epsilon^{\alpha\beta} E_\beta. \quad (26)$$

The expectation of the current

$$\langle j_\alpha(\mathbf{r}) \rangle = \frac{1}{Z} \int [d\phi] [da_\alpha] \left( \frac{e^2\pi}{h\theta} \epsilon^{\alpha\beta} E_\beta \right) e^{iS[a_\mu] + iS[\phi]}. \quad (27)$$

In the mean field theory, we can replace  $S[\phi] + S[a_\mu]$  by  $S$  which computed by using the classic path. Then the fields equate to their expectation value. The current equals to

$$\langle j_\alpha \rangle = j_\alpha = \frac{e^2\pi}{h\theta} \epsilon^{\alpha\beta} E_\beta. \quad (28)$$

Which means that the Hall conductance

$$\sigma_{xx} = 0, \sigma_{xy} = \frac{e^2}{h} \frac{1}{2k-1}. \quad (29)$$

Where  $k$  is an arbitrary integer, the Hall conductance is quantized. And the odd fraction should be the same as the “non-relativistic” electrons. Due to the Anderson-Higgs mechanism, the non-zero vacuum break the gauge symmetry results in the Meissner effect. The effect leads the state to be a incompressible quantum fluid, since any change of electron density will change the statistic gauge field which results in some flux that is forbidden by the Meissner effect.

## 4 The Vortices

Similar to the conventional electrons system, in addition to the uniform ground state there exist static, non-uniform, finite energy vortex solutions (topological excitations). The asymptotically behavior at  $(\mathbf{r} \rightarrow \infty)$  of the solution is

$$\phi(\mathbf{r}) = \left( \frac{\sqrt{\rho_1}}{\sqrt{\rho_2}} \right) e^{\pm i\psi(\mathbf{r})}, \quad (30)$$

$$\delta a = \mathbf{a} + \mathbf{A} = \pm \frac{1}{e} \nabla \psi(\mathbf{r}). \quad (31)$$

Where  $\psi$  is the angle of  $\mathbf{r}$ . Therefore for a large contour we have

$$\oint \delta a \cdot d\mathbf{l} = \pm 2\pi/e = \pm \phi_0 \quad (32)$$

Rather the quantization of flux, the charge is fractional quantized. To see this, form (12),

$$\rho = \bar{\rho} + \delta\rho = \frac{\nu}{\phi_0} \epsilon^{\alpha\beta} \partial_\alpha a_\beta = \frac{\nu}{\phi_0} \epsilon^{\alpha\beta} \partial_\alpha (\delta a - A_\beta) = \frac{\nu}{\phi_0} \epsilon^{\alpha\beta} \partial_\alpha \delta a_\beta + \frac{\nu}{\phi_0} B. \quad (33)$$

Therefor the excess charge of the vortex

$$Q = e \int d^2\mathbf{r} \delta\rho(\mathbf{r}) = e \frac{\nu}{\phi_0} \oint \delta a \cdot d\mathbf{l} = \pm e\nu. \quad (34)$$

This demonstrates that at the fractional filling the topological excitation have fractional charge. These field configurations corresponding to the quasiparticles and quasiholes above the ground state with fractional charge. The fractional charge implies that the qusiparticles and qusiholes are anyons obey fractional statistics with  $\theta_1 = \pi\nu$ . [15, 16, 17, 18, 19]

## 5 Conclusion

In this paper, we find that the statistic gauge transformation can also be applied to the interacting (pseudo) Dirac electrons system. In the Gindzburg-Landau-Chern-Simons theory proposed in ref. 5, the field coupled to the Maxwell field and the statistic gauge field is a bosonic complex scalar field, while in the (pseudo) Dirac electrons system, the bosonic field becomes a two dimensional complex scalar field. By considering the mean field solution of the effective field theory. We find similar results to “non-relativistic“ electrons: the fractional Hall conductance with odd denominators, the fractional charge of the quasiparticles. So we conclude that although the linear dispersion of (pseudo) Dirac electrons is different from that of the conventional electrons, the phenomenon of FQHE are also expected to be similar to the conventional electrons.

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